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LETTER TO THE EDITOR

The partition function for an anyon-like oscillator

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Abstract. We compute the partition function of an anyon-like harmonic oscillator. The well known results for both the bosonic and fermionic oscillators are then re-obtained as particular cases of our function. The technique we employ is a non-relativistic version of the Green function method used in the computation of one-loop effective actions of quantum field theory.

Partition functions give the statistical behaviour of a system of particles in thermal equilibrium with each other and with a thermal bath. In general, the search for these functions is not an easy task although for some particular systems there are well known results. Naturally, many techniques are available for performing this task. In particular, Gibbons [1] used the fact that the partition functions for the bosonic and fermionic oscillators could be written as determinants of the relevant operator, with periodic and antiperiodic boundary conditions, respectively, in order to compute them using the ζ-function method.

Specifically speaking, it follows directly from the definition of a partition function that, for any bosonic system, we can write

$$\begin{aligned} Z^{\mathrm{B}}(\beta) &= \mathrm{Tr}\, \mathrm{e}^{-\beta H} \\ &= \int \mathrm{d}x_0\, \langle x_0|\mathrm{e}^{-\beta H}|x_0\rangle \\ &= \int \mathrm{d}x_0\, K(x_0, x_0; \tau = -\mathrm{i}\hbar\beta) \end{aligned} \tag{1}$$

where $K(x, y; \tau)$ is the usual Feynman propagator. For the bosonic (harmonic) oscillator, we substitute its path integral representation and obtain

$$\begin{aligned} Z^{\mathrm{B}}(\beta) &= \int \mathrm{d}x_0 \int_{\substack{x(0)=x_0 \\ x(\beta)=x_0}} [Dx] \exp\left(-\tfrac{1}{2} \int_0^\beta x(\tau)(\omega^2 - \partial^2)x(\tau)\,\mathrm{d}\tau \right) \\ &= \int_{x(0)=x(\beta)} [Dx] \exp\left(-\tfrac{1}{2} \int_0^\beta x(\tau)(\omega^2 - \partial^2)x(\tau)\,\mathrm{d}\tau \right) \\ &= \det{}^{-1/2}(\omega^2 - \partial^2)|_{\text{periodic}}. \end{aligned} \tag{2}$$

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Analogously, using standard Grassmann variables, it can be shown that the partition function for the (second-order) fermionic oscillator is given by [1, 2]

$$Z^{\mathrm{F}}(\beta) = \det{}^{1}(\omega^2 - \partial^2)|_{\text{antiperiodic}}. \tag{3}$$

Here, we will in fact generalize these results by making analogous calculations, but this time we shall impose a generalized boundary condition which contains the particular cases of the periodic and antiperiodic conditions discussed in Gibbon's paper. Besides, we shall use an alternative technique which is the non-relativistic version of the Green function method for computing effective actions in quantum field theory [3, 4].

Suppose then that we want to compute the following determinant:

$$\exp[\Gamma_s^\theta(\omega)] = \det{}^{s}(\omega^2 + \partial_t^2)_\theta \equiv \det{}^{s}(L)_\theta \tag{4}$$

where L acts on functions that satisfy some given boundary condition specified by the label θ. The power of the determinant given by the parameter s is left completely arbitrary to take into account the cases that interpolate between the fermionic and the bosonic oscillators. Hence, playing with the boundary condition and the parameter s, we can pass continuously from the bosonic to the fermionic oscillator. This is why we refer to this determinant as the partition function of an 'anyon-like' oscillator.

Following a method usually employed in quantum field theory, we write

$$\begin{aligned}\frac{\partial}{\partial\omega}\Gamma_s^\theta(\omega) &= 2s\omega\,\mathrm{Tr}(L^{-1})_\theta \\ &= 2s\omega\int_0^\tau G_\omega^\theta(t,t)\,\mathrm{d}t\end{aligned} \tag{5}$$

where the Green function $G_\omega^\theta(t,t')$ satisfies

$$(\omega^2 + \partial_t^2)G_\omega^\theta(t,t') = \delta(t-t') \tag{6}$$

as well as some boundary condition (to be given in a moment). Integrating the above equation, we obtain

$$\Gamma_s^\theta(\omega) - \Gamma_s^\theta(0) = 2s\int_0^\omega \mathrm{d}\omega'\,\omega'\int_0^\tau \mathrm{d}t\,G_{\omega'}^\theta(t,t). \tag{7}$$

Since our purpose here is to use a generalized boundary condition, in the sense that the periodic and antiperiodic cases will appear as particular cases, and in order to make connection with the behaviour of correlation functions of anyon-like systems, it is natural to impose the following θ-dependent condition:

$$G_\omega^\theta(t+\tau,t') = \mathrm{e}^{-\mathrm{i}\theta}G_\omega^\theta(t,t'). \tag{8}$$

It is clear that this boundary condition becomes periodic for $\theta = 0$ and antiperiodic for $\theta = \pi$. Depending on these conditions and the value of the parameter s, which can be thought of as a 'statistical' parameter, this determinant will be mapped into different partition functions. As these particular cases are related to bosonic and fermionic systems, this condition of general periodicity could, in principle, be related to particles whose statistics interpolate bosons and fermions, i.e. anyons [5].

It is straightforward to construct the Green function $G^{\theta}_{\omega}(t-t')$. Using basically the same technique that Kleinert [6] employed for the simpler cases of periodic and antiperiodic boundary conditions, it can be shown that (see the appendix)

$$G^{\theta}_{\omega}(t-t') = \frac{\mathrm{e}^{-\mathrm{i}\theta/2}}{4\omega}\left[\frac{\mathrm{e}^{\mathrm{i}\omega(t-t'-\tau/2)}}{\sin(\frac{1}{2}\omega\tau+\theta)} + \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t'-\tau/2)}}{\sin(\frac{1}{2}\omega\tau-\theta)}\right] \qquad t-t' \in [0,\tau). \tag{9}$$

Substituting this Green function into equation (7), for the interval $[0,\tau)$ and with $t=t'$, we obtain

$$\begin{aligned}\Gamma^{\theta}_{s}(\omega) - \Gamma^{\theta}_{s}(0) &= 2s\int_0^{\tau}\mathrm{d}t\int_0^{\omega}\mathrm{d}\omega'\,\omega'\left\{\frac{\mathrm{e}^{-\mathrm{i}\theta/2}}{4\omega'}\left[\frac{\mathrm{e}^{-\mathrm{i}\omega'\tau/2}}{\sin(\frac{1}{2}\omega'\tau+\theta)} + \frac{\mathrm{e}^{\mathrm{i}\omega'\tau/2}}{\sin(\frac{1}{2}\omega'\tau-\theta)}\right]\right\} \\ &= \ln\left\{\mathrm{e}^{\mathrm{i}\theta}\left[-1+\mathrm{e}^{-\mathrm{i}(\theta+\omega\tau)}\right]\left[1-\mathrm{e}^{-\mathrm{i}(\theta-\omega\tau)}\right]\right\}^{s}. \end{aligned} \tag{10}$$

Recalling equation (4), we see that the exponential of $\Gamma^{\theta}_{s}(\omega)$ is the desired determinant. Identifying $\tau = -\mathrm{i}\beta$ ($\hbar = 1$), taking $\theta = 0$ (periodic boundary condition) and $s = -\frac{1}{2}$, this determinant reduces, apart from a constant factor $\exp[\Gamma^{\theta}_{s}(0)]$, which hereafter we call C, to the partition function for a bosonic oscillator [1, 6, 7], namely

$$Z^{\mathrm{B}}(\beta) = \exp[\Gamma^{0}_{-1/2}(\omega)] = \frac{C}{2\sinh(\omega\beta/2)}. \tag{11}$$

Note that any thermodynamical quantity which can be obtained from the partition function does not depend on C.

Analogously, for a fermionic oscillator, we just make $\theta = \pi$ (antiperiodic boundary condition) and $s = +1$, so that $\exp[\Gamma^{\theta}_{s}(\omega)]$ reduces to the following partition function

$$Z^{F(2)}(\beta) = \exp[\Gamma^{\pi}_{+1}(\omega)] = 4C\cosh^2(\omega\beta/2). \tag{12}$$

This result corresponds to the partition function for a second-order fermionic oscillator. One can check this by explicitly calculating the eigenvalues for the Finkelstein–Villasante Grassmann oscillator (with $N=2$) [8] and then finding its partition function by summing the trace of $\exp(-\beta E_n)$. This result differs from the one given by Gibbons [1] in the quadratic power of $\cosh(\omega\beta/2)$, once he considered an equivalent linear Grassmann oscillator opposed to the quadratic case discussed here. This linear case can also be obtained from our discussion if we take, from the very beginning, the determinant of $L^{1/2}$ instead of L, so that we find

$$Z^{\mathrm{F}(1)}(\beta) = \exp\left[\Gamma^{\pi}_{+1}(\omega)\right]_{\mathrm{Linear}} = 2C\cosh(\omega\beta/2) \tag{13}$$

which is the well known partition function for the linear fermionic oscillator [1, 6, 7].

For the general case, the partition function reads:

$$\begin{aligned} Z^{\theta}_{s}(\beta) &\equiv \exp\left[\Gamma^{\theta}_{s}(\omega)\right] \\ &= \left[\mathrm{e}^{-\mathrm{i}\theta}\left(-1+\mathrm{e}^{-\mathrm{i}(\theta+\omega\tau)}\right)\left(1-\mathrm{e}^{-\mathrm{i}(\theta-\omega\tau)}\right)\right]^{s} \\ &= 4^{s}[\cosh^2\tfrac{1}{2}\omega\beta - \cos^2\tfrac{1}{2}\theta]^{s} \end{aligned} \tag{14}$$

where for simplicity we put $C=1$. Naturally, the above calculated partition functions are particular cases of equation (14).

Note that, in equation (14), we have left the statistics parameter s free. In fact, it may be a function of the periodicity parameter θ, interpolating between $s(\theta = 0) = -\frac{1}{2}$ (bosonic case) and $s(\theta = \pi) = +1$ (fermionic case), as for example $s(\theta) = -\frac{1}{2} + \frac{3}{2} f(\theta)$, where $f(\theta)$ may be a function which satisfies $f(0) = 0$ and $f(\pi) = +1$. We wonder whether this factor can be obtained from the functional integration of a generalized variable, with an arbitrary commutation relation, interpolating the cases of bosonic (c-number) and Grassmannian variables, as a kind of q-deformed calculation [9, 10].

Another possible interpretation for this generalized partition function is in relation to parasystems [11], where there is a parameter for which convenient limits reproduce the bosonic and fermionic oscillators [12]. However, the connection between these systems with the present calculations (if any) seems to be non-trivial and deserves further study. This will be discussed elsewhere.

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Appendix

In this appendix we construct the Green function (9). The following discussion is similar to that found in Kleinert [6]. In fact, we generalize Kleinert's calculations for the cases where the boundary conditions are neither periodic nor antiperiodic, but anyon-like.

The spectral representation for $G^\theta_\omega(t)$ is given by

$$
\begin{aligned}
G^\theta_\omega(t) &= \frac{1}{\tau} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega^\theta_m t}}{\omega^2 - {\omega^\theta_m}^2} \\
&= \frac{1}{\tau} \sum_{m=-\infty}^{\infty} \left\{ \frac{\mathrm{e}^{-\mathrm{i}\omega^\theta_m t}}{2\mathrm{i}\omega} \left[\frac{\mathrm{i}}{\omega - \omega^\theta_m} + \frac{\mathrm{i}}{\omega + \omega^\theta_m} \right] \right\} \\
&= \frac{1}{2\omega\mathrm{i}} \{ G^\theta_-(t) + G^\theta_+(t) \}
\end{aligned} \tag{15}
$$

where $\omega^\theta_m = (2\pi m + \theta)/\tau$ and we have identified the spectral representations of $G^\theta_\pm(t)$, which are, respectively, the Green functions associated with the first-order operators $(\mathrm{i}\partial_t \pm \omega)$ also satisfying the generalized periodic boundary condition (8).

Using the Poisson summation formula [13]

$$
\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} \mathrm{d}\mu \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i}\mu n} f(\mu) \tag{16}
$$

we can write for $G^\theta_\pm(t)$

$$
\begin{aligned}
G^\theta_\pm(t) &= \frac{\mathrm{i}}{\tau} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega^\theta_m t}}{\omega \pm \omega^\theta_m} \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega' t - n\omega'\tau + n\theta)} \left(\frac{\mathrm{i}}{\omega \pm \omega'} \right).
\end{aligned} \tag{17}
$$

Identifying

$$G_\pm(t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega' t} \left(\frac{i}{\omega \pm \omega'} \right) \tag{18}$$

as the Green functions associated with the first-order operators $(i\partial_t \pm \omega)$, but this time valid for an infinite time interval, we may cast (17) in the form

$$G^\theta_\pm(t) = \sum_{n=-\infty}^{\infty} e^{-in\theta} G_\pm(t - n\tau). \tag{19}$$

Hence, to obtain $G^\theta_\pm(t)$, we need first to compute $G_\pm(t)$. By residue calculations this can be made after setting $\omega \to \omega - i\eta$. With this prescription, it can easily be shown that

$$G_\pm(t) = -\Theta(\mp t) e^{\pm i\omega t} \tag{20}$$

where $\Theta(t)$ is the usual Heaviside step function. Substituting (20) into (19), we obtain

$$G^\theta_\pm(t) = -\sum_{n=-\infty}^{\infty} e^{-in\theta} \Theta(\mp t \pm n\tau) e^{\pm i\omega(t - n\tau)}. \tag{21}$$

Since this expression has a period τ, we can restrict ourselves to the interval $t \in [0, \tau)$. Hence, the sum appearing in the right-hand side of (21) can be obtained easily, yielding, for $G^\theta_+(t)$,

$$\begin{aligned} G^\theta_+(t) &= -e^{i\omega t} \sum_{n=1}^{+\infty} e^{-in(\omega\tau + \theta)} \\ &= -e^{i\omega t} \{ e^{-i(\omega\tau+\theta)} + e^{-2i(\omega\tau+\theta)} + \cdots \} \\ &= -\frac{e^{i\omega t} e^{-i(\omega\tau+\theta)}}{1 - e^{-i(\omega\tau+\theta)}}. \end{aligned} \tag{22}$$

Analogously, for $G^\theta_-(t)$, we have

$$\begin{aligned} G^\theta_-(t) &= -e^{-i\omega t} \sum_{n=0}^{-\infty} e^{-in(\omega\tau - \theta)} \\ &= -e^{-i\omega t} \{ 1 + e^{-i(\omega\tau-\theta)} + e^{-2i(\omega\tau-\theta)} + \cdots \} \\ &= -\frac{e^{-i\omega t}}{1 - e^{-i(\omega\tau-\theta)}}. \end{aligned} \tag{23}$$

Of course, outside this interval, the result can be obtained by periodicity. Substituting (22) and (23) into (15) and rewriting t as $t - t'$, we finally obtain

$$G^\theta_\omega(t - t') = \frac{e^{-i\theta/2}}{4\omega} \left[\frac{e^{i\omega(t - t' - \tau/2)}}{\sin(\frac{1}{2}\omega\tau + \theta)} + \frac{e^{-i\omega(t - t' - \tau/2)}}{\sin(\frac{1}{2}\omega\tau - \theta)} \right] \qquad t - t' \in [0, \tau). \tag{24}$$

This Green function satisfies equations (6) and (8). This formula generalizes the previously known particular cases of periodic ($\theta = 0$) and antiperiodic ($\theta = \pi$) boundary conditions, for which we find the well known results in accordance with the literature [6].

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